Maximization and logarithmic transformation

Given the function $f(x, y) = xye^{-(x+y)}$. With $x, y > 0$, it is requested:

1. Find the value (x, y) that maximizes f.
2. Now consider the problem of maximizing the function $g(x, y) = ln(f(x, y))$. What can you observe? Draw conclusions and justify.

Solution

1. We calculate the first order conditions:

$$f'_x = ye^{-(x+y)} + xye^{-(x+y)}(-1) = 0$$

$$f'_y = xe^{-(x+y)} + xye^{-(x+y)}(-1) = 0$$

We divide both equations by $e^{-(x+y)}$ on both sides:

$$f'_x = y - xy = 0$$

$$f'_x = x - xy = 0$$

Rearranging,

$$y = xy$$

$$x = xy$$

Since we know that $x > 0$ and $y > 0$ we clear the first equation:

$$1 = x^*$$

Hence from the second equation:

$$1 = y^*$$

With this we have the following two points: $(0, 0)$ and $(1, 1)$. Now let's move to the second order conditions by calculating the second derivatives.

2. Maximizing the function and maximizing the natural logarithm of the function will result in the same points, this is because strictly increasing transformations do not alter the critical points: $g(x, y) = ln(xye^{-(x+y)}) = ln(x) + ln(y) - x - y$.

$$g'_x = 1/x - 1 = 0$$

$$g'_y = 1/y - 1 = 0$$

From here we obtain that:

$$x^* = 1$$

$$y^* = 1$$

Now let's check the second order conditions by building the hessian matrix:

$$f''_{xx} = -1/x^2$$

$$f''_{yy} = -1/y^2$$

$$f''_{xy} = f''_{yx}$$

$$H = \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{pmatrix} = \begin{pmatrix} -1/x^2 & 0 \\ 0 & -1/y^2 \end{pmatrix}$$

Evaluating at the critical point:

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

The determinant is $1 > 0$, and since we have $f''_{xx} < 0$ we are facing a relative maximum.